Section 1a:
Number Skills

Maths Literacy Module, Workshop Series 2010

Section 1a: Number Skills

1. Introduction

This section on number skills lays the foundation for the other sections, as it provides the student with the opportunity to review basic mathematical concepts that will be used in this subject throughout the year. As the student, you need to make sure that on completion of this section you are comfortable working with all of its contents.

Learning outcomes

At the end of this section on Number Skills you should be able to perform basic calculations confidently.

START UP ACTIVITY 1.1:
What do I know?

Pair up with a class mate and complete the following activity within 15 minutes:

Try to define each of the concepts listed below and support your definition with an example in each case. Then share your findings with the rest of the class.

- Fractions
- Decimals
- Rounding off
- Percentages
- Ratio and Proportion
- Sigma notation $\sum$
- Exponents
- Logarithms

2. Fractions

Most of us are accustomed to fractions such as "three over four", $\frac{3}{4}$, and would interpret this specific fraction as " three parts of four" or "three divided by four". This is indeed correct. However, fractions are also known as rational numbers and the set of all rational numbers is denoted by the symbol $\mathbb{Q}$.

Definition of a Rational Number

A *rational number* is a number that can be written as a fraction $\frac{a}{b}$, where a and b are integers and $b \neq 0$. (Note that division by 0 is not defined.)

We can also refer to $\frac{a}{b}$ as the **quotient** of a and b, where a is called the **numerator** and b the **denominator** or **divisor** of the quotient.

EXAMPLE 1.1

$\frac{5}{7}, \frac{20}{-9}, \frac{0}{6}, \frac{4}{1}$ and $\frac{-3}{8}$ are all examples of rational numbers.

To perform different operations such as addition, multiplication and division with rational numbers (fractions), we need certain properties of fractions that we can use to facilitate matters for us.

Properties of Fractions

1 $\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$

To multiply fractions, we multiply the numerators with each other (to form the numerator of the product) and the denominators with each other (to form the denominator of the product).

EXAMPLE 1.2

$$\frac{3}{5} \times \frac{7}{2} = \frac{3 \times 7}{5 \times 2} = \frac{21}{10}$$

2 $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$

To divide fractions, invert the divisor and multiply it with the first fraction. This operation becomes meaningful if we look at the following explanation:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b} \times \frac{d}{c}}{\frac{c}{d} \times \frac{d}{c}} = \frac{\frac{ad}{bc}}{\frac{cd}{dc}} = \frac{\frac{ad}{bc}}{1} = \frac{ad}{bc}$$

EXAMPLE 1.3

$$\frac{3}{5} \div \frac{7}{2} = \frac{3}{5} \times \frac{2}{7} = \frac{6}{35}$$

Please note that we are NOT doing cross-multiplication here. Cross-multiplication is a method of simplifying fractions in an equation. Recall that an expression is obtained by combining variables and constants by using operations such as addition, subtraction, multiplication, division and taking roots. For example, $2x - \frac{1}{3}x^2 + 7 + \sqrt{4x}$ is an expression. An equation is a statement that put two mathematical expressions equal to each other. For example, $7 - 4 = 3$ and $2x + 4 = 9$ are equations.

3 $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

To add fractions that have the same denominators, you only need to add the numerators (to form the numerator of the sum), and the denominator stays the same.

EXAMPLE 1.4

$$\frac{3}{5} + \frac{6}{5} = \frac{3+6}{5} = \frac{9}{5}$$

4 $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

To add fractions with different denominators, find a common denominator and rewrite the terms in the numerator with respect to the new denominator. Then apply 3, i.e., add the two new numerators to obtain the numerator of the sum, and keep the denominator of the sum as the common denominator of the two fractions. This process once again becomes meaningful if we look at the following explanation:

$$\frac{a}{b} + \frac{c}{d} = \frac{a \times d}{b \times d} + \frac{c \times b}{d \times b} = \frac{ad}{bd} + \frac{bc}{bd} = \frac{ad+bc}{bd}$$

EXAMPLE 1.5

$$\frac{3}{5} + \frac{7}{2} = \frac{3 \times 2}{5 \times 2} + \frac{7 \times 5}{2 \times 5} = \frac{6+35}{10} = \frac{41}{10}$$

Please note that we are once again NOT doing cross-multiplication here. Cross-multiplication is an operation which simplifies fractions in an equation.

5 $\frac{ac}{bc} = \frac{a}{b}$

Factors that are common in the numerator and the denominator can be cancelled out.

Example 1.6

$$\frac{3\times 4}{5\times 4}=\frac{3}{5}$$

6 $\frac{a}{b}=\frac{c}{d}$ if and only if $ad=bc$

Here we use cross-multiplication, as we are working with an equation. What actually happens, is that both sides of the equation are multiplied with the common denominator bd : $\frac{a\times bd}{b}=\frac{c\times bd}{d}$, hence $ad=bc$. If $ad=bc$, then we divide both sides by bd to see that $\frac{a}{b}=\frac{c}{d}$.

Example 1.7

$\frac{3}{5}=\frac{9}{15}$, because $3\times 15=9\times 5$

Example 1.8

Evaluate $\frac{9}{24}+\frac{211}{300}$

Solution

When evaluating a sum of fractions for which a common denominator can be found which is smaller than the product of the two denominators, as described in property 4 above, the following procedure is used. Each denominator is factored into its prime factors (a prime factor has only 1 and itself as positive divisors, such as 2, 3, 5, 7,…). So $24=2^3\cdot 3$ and $300=2^2\cdot 3\cdot 5^2$. The **Least Common Denominator** (LCD) is then determined by forming the product of all the factors that occur in the factorizations, using the highest power of each factor. Therefore the LCD of 24 and 300 is $2^3\cdot 3\cdot 5^2=600$.

Then

$$\begin{aligned}\frac{9}{24}+\frac{211}{300} &= \frac{9\times 25}{24\times 25}+\frac{211\times 2}{300\times 2}\\ &= \frac{225}{600}+\frac{422}{600}\\ &= \frac{647}{600}\end{aligned}$$

LEARNING ACTIVITY 1.2

Simplify the following fractions as far as possible:

1. $\frac{7}{9} - \frac{4}{9}$
2. $\frac{7}{9} \times \frac{4}{9}$
3. $\frac{7}{9} \div \frac{4}{9}$
4. $\frac{3}{4} + \frac{4}{5}$
5. $\frac{5}{12} - \frac{9}{20}$

3. Decimals

It is not always easy and practical to work with numbers in fractional form. You would, for instance, be very surprised to see the price of a pair of jeans being advertised as $R129\frac{95}{100}$. Or a packet of biltong being sold for $R\frac{75}{4}$. We therefore often choose to express fractions as decimals.

All fractions (rational numbers) can be expressed as recurring decimals, that is, a decimal in which one or more of the digits are ultimately repeated indefinitely. We write a recurring decimal by placing a dot or a line over the digit or digits that repeat. Here are some examples of recurring decimals:

$$\frac{1}{3} = 0{,}333333\ldots = 0{,}\dot{3}$$
$$\frac{157}{495} = 0{,}317171717\ldots = 0{,}3\overline{17}$$
$$\frac{9}{7} = 1{,}285714285714285714\ldots = 1{,}\overline{285714}$$
$$\frac{1}{2} = 0{,}50000\ldots = 0{,}5\dot{0}$$

The last example is also called a terminating decimal, as it can be written as a fraction with a denominator that can be expressed as a power of 10. Here we have that $\frac{1}{2} = \frac{5}{10} = 0{,}50000\ldots = 0{,}5\dot{0}$. It is customary to omit recurring zeros, so we simply write $\frac{1}{2} = 0{,}5$ instead of $\frac{1}{2} = 0{,}5\dot{0}$.

Whenever we encounter a recurring or terminating decimal, we know that it can be expressed as a fraction. To convert a recurring decimal, which is sometimes also call a repeating decimal, such as $x = 2{,}578787878\ldots = 2{,}5\overline{78}$ to rational (or fractional) form, we follow the following procedure:

Step 1 Multiply the decimal number with a suitable power of 10 so that the comma moves to the right enough positions to just past the first recurrence. For example, if $x = 2{,}578787878\ldots = 2{,}5\overline{78}$, we multiply x by 1000 to move the comma three positions to the right until just after the first digit 8, i.e., until after the first occurrence of the repeating part 78:

$$1000x = 2578{,}7878\ldots$$

Step 2 Multiply the decimal number with a suitable power of 10 so that the comma moves to the right enough positions until just before the first occurrence of the repeating part. In the example above we multiply x by 10 to move the comma until just before the first occurrence of the repeating part 78.

$$10x = 25{,}787878\ldots$$

Step 3 Now subtract the two equations obtained in steps 1 and 2 from each other in order to eliminate the decimal part.

$$\begin{array}{rcl} 1000x & = & 2578{,}787878\ldots \\ 10x & = & 25{,}787878\ldots \\ \hline 990x & = & 2553{,}0 \\ \hline \end{array}$$

Step 4 Divide by the coefficient of x (990 in our example) on both sides of the last equation to express x as fraction. In our example, $x = 2{,}5\overline{78} = \dfrac{2553}{990}$.

LEARNING ACTIVITY 1.3

1. Write the following fractions as decimals by using your calculator. Indicate the recurring part in the decimal(s) with a dot or a line:

1.1. $\dfrac{221}{90}$

1.2. $\dfrac{1959}{990}$

1.3. $\dfrac{7}{33}$

2. Convert the recurring decimal $5{,}92768768768\ldots = 5{,}92\overline{768}$ to a fraction.

4. Rounding off

When we round off a number, we actually give an approximate value of the number. It is usually easier to calculate with the approximate value. For example, the number $6{,}8$ rounded off to the nearest whole number (integer) becomes 7. But $6{,}8$ is clearly not equal to 7. 7 is just a useful approximation of 6,8 (it is a rough indication of the number itself). Let us consider the rules that are used when rounding off numbers written in decimal form.

When rounding off a decimal number to the nearest whole number, we consider the first digit after the comma. If the first decimal is equal to or greater than five, the number is rounded up to the first whole number greater than the decimal number. For example, $5{,}7$ is rounded to 6. If the first digit after the comma is smaller than five, the decimal number is rounded off to the whole number that appears to the left of the comma. For example, $6{,}4$ is rounded off to 6.

When rounding off a decimal number to the first decimal, we consider the second digit after the comma. If the second decimal is equal to or greater than five, the first digit after the comma is increased by one and the second digit after the comma becomes zero (or falls away). For example, $5{,}75$ is rounded to $5{,}8$. If the second digit after the comma is smaller than five, the first digit after the comma keeps its value and the second decimal becomes zero (or falls away). For example, $6{,}53$ is rounded to $6{,}5$.

When rounding off a decimal value to the second decimal, we consider the third digit after the comma. If the third digit is equal to or greater than five, the second decimal after the comma is increased by one and the third digit after the comma becomes zero (or falls away). For example, $5{,}756$ is rounded to $5{,}76$. If the third digit after the comma is smaller than five, the second digit after the comma keeps its value and the third digits becomes zero (or falls away). For example, $6{,}532$ becomes $6{,}53$.

These are the basic rules to follow when rounding off decimal numbers. In a similar way decimal numbers can be rounded off to any number of decimals places after the comma.

EXAMPLE 1.9

1 $98{,}63975$ becomes $98{,}6398$ if it is rounded off to four decimal places
2 $98{,}63975$ becomes $98{,}6$ if it is rounded off to one decimal place

LEARNING ACTIVITY 1.4

1. Round the number $6{,}9102837465$ off to the indicated number of decimal places:

1.1. The nearest whole number
1.2. One decimal place
1.3. Two decimal places
1.4. Three decimal places
1.5. Four decimal places
1.6. Five decimal places
1.7. Six decimal places
1.8. Seven decimal places
1.9. Eight decimal places
1.10. Nine decimal places

2. Which of your answers above is the closest to the original value $6{,}9102837465$? Give a reason for your answer.

5. Percentages

Percentages form part of our everyday lives. They are often quoted in the media, particularly in connection with money matters:

'Inflation now stands at 7,5%.''Ford has given its workforce a 24,2% raise.''Unemployment in South Africa is about 36%.'

In the sections on fractions and decimals, we learned some very useful ways to express parts (fractions) of quantities and how to compare these. In this section we now standardise our comparison of parts of quantities. We do this by expressing quantities in relation to a standard unit of 100. We use this relationship, called **percentage**, to solve various types of problems.

The word percent means "per hundred" or "for every hundred". So, if you pay 14 percent VAT (Value Added Tax) on an item, it means that for every R100 of the value of the item, you have to pay R14 VAT. The symbol % represents "percentage".

Definition of a Percent

A percent is a fractional part of a hundred, expressed by the symbol %.

A hundred percent (100%) represents one whole quantity, or one hundred out of a hundred parts, $\left(\frac{100}{100}\right)$. Using the label "percent" is an easy way to express the relationship of any quantity with respect to 100. When we want to make a comparison or state a problem, the term percent may save time in performing a calculation. But we can't use percentages as such when solving a problem. First of all, we need to convert the percentage to a fraction or a decimal.

To change a percentage such as 45% to a numerical equivalent (its fractional form), we need to divide by 100 (since 45% means 45 out of every 100). Therefore, for every real number a, $a\%$ is numerically equal to $a \div 100$ or $\frac{a}{100}$. So $45\% = \frac{45}{100}$.

Percentages often help us make comparisons between numbers.

EXAMPLE 1.10

Suppose a student obtained the following marks in a series of tests:

Mathematics	English	Life Skills
$\frac{16}{25}$	$\frac{31}{50}$	$\frac{30}{40}$

It is difficult to compare these results, as the tests were not out of the same mark. The student obtained the most marks in English, but was this his best result? In order to compare the marks, the fractions should all be converted to percentages, which will give us results expressed as fractions with the same denominator, namely 100. This is done as follows:

$$\frac{16}{25}\times\frac{4}{4}=\frac{64}{100}=64\% \qquad \frac{31}{50}\times\frac{2}{2}=\frac{62}{100}=62\% \qquad \frac{30}{40}\times\frac{2,5}{2,5}=\frac{75}{100}=75\%$$

The test results are therefore 64% in Mathematics, 62% in English and 75% in Life Skills. The best result was obtained in Life Skills.

To convert a fraction or decimal to a percentage, just multiply it by 100.

EXAMPLES 1.11

Expressing a decimal as a percentage

1 $0,59\times 100=59\%$

2 $1,45\times 100=145\%$

Expressing a fraction as a percentage

3 $\frac{34}{100}\times\frac{100}{1}=34\%$

4 $\frac{103}{100}\times\frac{100}{1}=103\%$

5 $\frac{19}{36}\times\frac{100}{1}=\frac{1900}{36}=52,78\%$

Expressing a percentage as a decimal and a fraction

6 $30\%=0,30=\frac{30}{100}$

Increasing an Amount

In real life it is often indicated that the price of products will rise by a certain percentage. We could hear on the news that bread, for example, will increase in price by 3% as from next week. What does such a percentage increase mean in cash terms?

EXAMPLE 1.12

If bread currently costs $R7,50$ and increases in price by 3%, how much will it cost after the raise?

SOLUTION

There are two ways in which we can determine the new price of bread. Firstly, we can calculate 3% of $R7{,}50$ and then add it to the original price:

3% of $R7{,}50$ $= \frac{3}{100} \times R7{,}50$

$= 23c$

New price $= R7{,}50 + R0{,}23$

$= R7{,}73$

Secondly, we can consider the original amount of $R7{,}50$ as 100% of the price. Increasing it by 3% is the same as finding 103% of the original price.

New cost $= 103\% \times R7{,}50$

$= \frac{103}{100} \times R7{,}50$

$= R7{,}73$

Decreasing an Amount

Just as prices of items may rise occasionally, they do sometimes become cheaper. The amount of money that we pay for these items then decreases.

EXAMPLE 1.13

The price of a movie ticket is $R23{,}50$. What would the price be after the discount as indicated on the banner is applied?

SOLUTION

There are two ways in which we can determine the new price of a movie ticket. Firstly, we can calculate 5% of $R23{,}50$ and then subtract it from the original price:

5% of $R23{,}50$ $= \frac{5}{100} \times R23{,}50$

$= R1{,}18$

New price $= R23{,}50 - R1{,}18$

$= R22{,}32$

Secondly, we can consider the original amount, namely $R23{,}50$, as 100% of the price. Decreasing it by 5% is the same as finding 95% of the original price.

New cost $= 95\% \times R23{,}50$

$= \frac{95}{100} \times R23{,}50$

$= R22{,}32$

Percentage increase or decrease

To find the percentage increase / decrease in the price of an item, we can make use of the following formula.

Percentage increase/decrease

Percentage increase / decrease $= \dfrac{\text{New amount - Original amount}}{\text{Original amount}} \times 100$

Example 1.14

You earn R4 000 pm. If your salary increases to R4 300 pm, determine the percentage of increase.

Solution

$$\textbf{Percentage increase} = \frac{\text{New salary - Original salary}}{\text{Original salary}} \times 100$$
$$= \frac{R4300 - R4000}{R4000} \times 100$$
$$= 7{,}5\%$$

The percentage increase is 7,5%.

Example 1.15

If the price of petrol decreases from R5,20 to R4,80 per litre, determine the percentage of decrease.

Solution:

$$\textbf{Percentage decrease} = \frac{\text{New price - Original price}}{\text{Original price}} \times 100$$
$$= \frac{R4,80 - R5,20}{R5,20} \times 100$$
$$= -7{,}7\%$$

The percentage decrease is 7,7%.

LEARNING ACTIVITY 1.5:

1. Express the value $0{,}456$ as a percentage.
2. Use your calculator to express the fraction $\frac{13}{17}$ as a percentage and round your answer off to two decimal places.
3. Express 38% as a decimal and as a fraction out of 100.
4. zYour sister is struggling with her grade 11 school work and got the following marks in three different tests:
   4.1. $\frac{14}{23}$ for her mathematics test
   4.2. $\frac{12}{30}$ for her biology test
   4.3. $\frac{15}{41}$ for her physics test

   Which subject does she struggle with most according to these scores? Use you calculator to help you with this question.
5. A furniture store has a sale on lounge suites. A certain suite usually costs $R11999$, but is now on promotion and will be sold for 15% less. Your mother is convinced that your father has to buy it for her as an anniversary gift. How much will your father pay for the lounge suite?
6. If a consultation fee increases from R350 to R380 per hour, determine the percentage increase in the consultation fee.
7. Calculate the inflation rate as a percentage if the price of petrol rises from R4,20 to R4,95 per litre.
8. The price of product A increases from R10 to R20 and the price of product B increases from R40 to R60. Which product had the largest percentage increase?

6. Ratio and Proportion

Ratio

Definition of Ratio

A ratio compares two or more quantities of the same kind, expressed in the same unit.

EXAMPLE 1.16

At a BMW dealership there are 24 blue BMWs for sale and 15 silver ones. The ratio of blue to silver BMWs can be written as $24:15$. Note that the unit in this example is simply "cars" and is not included or shown in the ratio.

A ratio can be simplified by dividing each side of the ratio by the same number.

EXAMPLE 1.17

Cost price : Selling price	$= 250:300$	*(No units are shown, but have to be the same – rands on both sides)*
	$= 10:12$	*(Divide both sides by 25)*
	$= 5:6$	*(Divide both sides again – now by 2 – to obtained the simplest form)*

Quantities can be divided into a certain ratio, as illustrated in the following example.

EXAMPLE 1.18

Mr Jones, Mrs Black and Dr Edwards invested in an investment opportunity through a bank and made a total profit of R1 048 238. Mr Jones invested R180 000, Mrs Black invested R315 000 and Dr Edwards invested R135 000. Divide the profit among the investors in the same ratio as their investments.

SOLUTION

First we need to determine the ratio between the amounts that they have invested.

Jones : Black : Edwards = 180 000 : 315 000 : 135 000
= 4 : 7 : 3 (after simplification)

We get this simplified ratio by dividing each amount by 45 000.
The sum of the parts is 4 + 7 + 3 =14.

Therefore: Mr Jones receives $\frac{4}{14}$ × R1 048 238 = R299 496,57

Mrs Black receives $\frac{7}{14} \times$ R1 048 238 = R524 119,00

Mr Edwards receives $\frac{3}{14} \times$ R1 048 238 = R224 622,43

Proportion

Definition of Direct Proportion

Two quantities are in direct proportion if the one quantity increases if and only if the other quantity increases.

These two quantities, which we can be represent by x and y, are in a functional relationship that can be presented in two different ways:

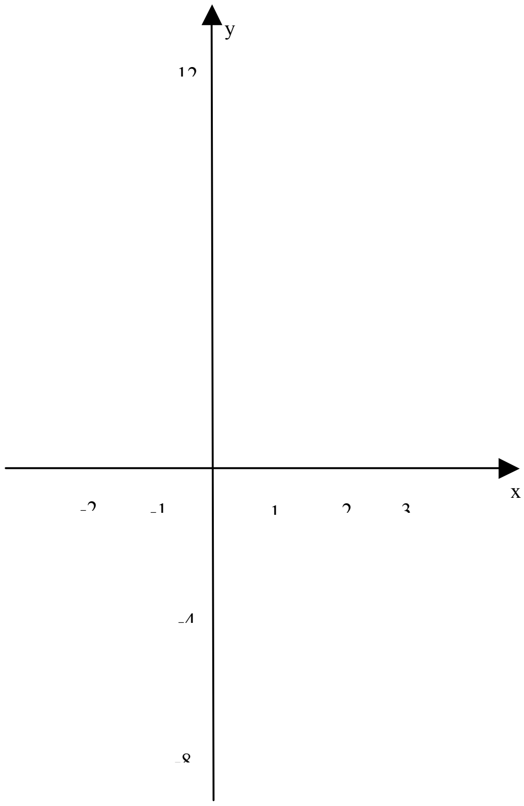
1 With an **equation** of the form $y = mx$, where m is positive, such as $y = 4x$. Then:

$x =$	$y =$
-3	-12
-2	-8
-1	-4
0	0
1	4
2	8
3	12

Here we can clearly see how the $y-$ values increase as the $x-$ values increase.

Or

2 With a **straight line graph** going through the origin (0, 0)



This straight line graph represents the equation $y = 4x$.

EXAMPLE 1.19

4 kg of potatoes costs $R12,50$. How much does 11 kg cost?

SOLUTION

If 4 kg costs $R12,50$, then

$\frac{4}{4} = 1$ kg costs $\frac{12,50}{4} = R3,13$.

Therefore 11 kg of potatoes costs $11 \times R3,13 = R34,43$

EXAMPLE 1.20

On Sunday it will be your 21st birthday and your father promised to take you and a few friends for lunch to an exclusive restaurant. The restaurant charges $R150$ per person for a full-course meal including two drinks per person. Initially you decided to take only your three best friends to the lunch, but your dad said that he can spend $R1350$ on the lunch. How many more friends can you now take to the lunch? Your father will not join you for the lunch.

SOLUTION

If you take only your three best friends to the lunch, you will be four people (including yourself) and it will cost your father $4 \times R150 = R600$.

The available $R1350$ will cover the cost of $\frac{1350}{150} = 9$ people. You can therefore invite another $9 - 4 = 5$ friends to the lunch.

Definition of Inverse Proportion

Two quantities are in inverse proportion if the one quantity increases if and only if the other quantity decreases.

These two quantities, which we can be represented by x and y, are in a functional relationship to each other that can be presented in two ways:

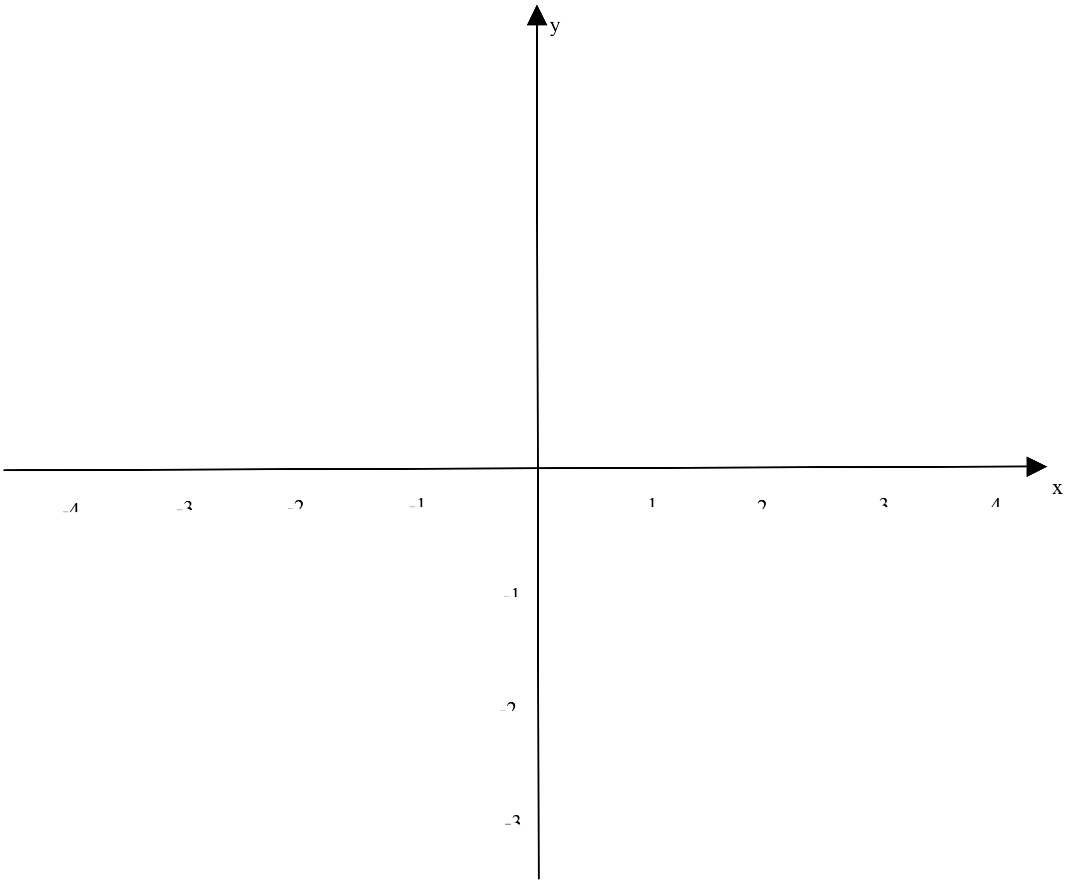
1 With an **equation** of the form $xy = k$ (or $y = \frac{k}{x}$), where k is positive.

If $y = \frac{3}{x}$, then we have that

$x =$	$y =$	
-4	$-\frac{3}{4}$	
-3	$-\frac{3}{3} = -1$	
-2	$-\frac{3}{2}$	Here we can clearly see how the $y-$values decrease
-1	$-\frac{3}{1} = -3$	from $-\frac{3}{4}$ to -3 as the $x-$values increase from -4 to -1.
0	(Undefined)	Division by 0 is not defined.
1	$\frac{3}{1} = 3$	
2	$\frac{3}{2}$	It is also clear that the $y-$values decrease from 3 to $\frac{3}{4}$
3	$\frac{3}{3} = 1$	as the $x-$values increase from 1 to 4.
4	$\frac{3}{4}$	

or

2 With a **graph of a hyperbola** with center at (0, 0)



This hyperbola represents the equation $y = \frac{3}{x}$.

EXAMPLE 1.21

9 workers were asked to paint the walls on the inside of an apartment building and they took 2 weeks to complete it. How many days would it take to complete this job if 15 workers were used instead? (Assume that all workers always work at the same rate.)

SOLUTION

This is an inverse proportion problem as the number of days that it would take to paint the building will surely be less if more painters are employed to do the job. We will therefore use the equation $xy = k$ and first determine the constant k.

The given values of x and y are:
$x = 9$ workers took $y = 14$ days
$\therefore k = xy = 9 \times 14 = 126$

If the number of workers (x) increases to 15, we then have that

$$y = \frac{k}{x}$$
$$= \frac{126}{15}$$
$= 8{,}4$ days to complete the job.

LEARNING ACTIVITY 1.6

1. Write a simplified ratio for the number of oranges to the number of apples sold at a fruit and vegetable store on a specific day, if 45 oranges and 39 apples were sold.

2. A certain bank manager had three daughters aged 16, 13 and 8. He saved $R23589$ over the past few years and wanted to use the savings for his daughters' school expenses the following year. If the money is to be shared between the girls according to their ages, how much money will be spent on each girl the following year?

3. I paid $R25$ for three hot-cross buns at the local bakery yesterday morning. They were so delicious that I want to buy eight hot-cross buns today to share among my three friends and myself. How much money do I need to buy the eight buns?

4. 12 workers are instructed to lay carpets in a townhouse complex that has 42 units in it. They took 25 days to complete the job.
   4.1. How many days would it take to lay the carpets if 15 workers were used instead?
   4.2. How many days less (than the 12 workers) did the 15 workers work?

7. Sigma Notation

The sigma notation is used to write certain lengthy sums (or series) in a compact form. The sums that can be written in this way are special. The terms of such a series need to have a general pattern and must appear in a specific order. The symbol Σ used in sigma notation is the Greek capital letter S that we use to indicate a "sum".

If we want to add the terms $a_1, a_2, a_3, \ldots, a_n$, sigma notation can be used to express the sum $a_1 + a_2 + a_3 + \cdots + a_n$ compactly as $\sum_{k=1}^{n} a_k$.

The letter k is called the **index of summation** and a_k is the general term of the sum. The idea is to substitute successively the values of k into the general term a_k, starting with $k = 1$ and terminating at $k = n$, and then add all these values.

EXAMPLE 1.22

To find the first three terms, the $20th$ term and the $598th$ term of the sequence defined by the general term $a_k = 2k - 15$, we have to substitute k=1, 2, 3, 20 and 598 into the formula representing the general term. Then we get that
the first term $a_1 = 2(1) - 15 = -13$,
the second term $a_2 = 2(2) - 15 = -11$,
the third term $a_3 = 2(3) - 15 = -9$,
the twentieth term $a_{20} = 2(20) - 15 = 25$ and
the five hundred and ninety eighth term $a_{598} = 2(598) - 15 = 1181$.

If the sum of a number of terms is expressed in sigma notation, we can determine the sum as demonstrated in the following three examples.

EXAMPLE 1.23

Determine the sum: $\sum_{k=1}^{7} k^2$.

SOLUTION

$$\begin{aligned}\sum_{k=1}^{7} k^2 &= 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 \\ &= 1 + 4 + 9 + 16 + 25 + 36 + 49 \\ &= 140\end{aligned}$$

Example 1.24

Determine the sum: $\sum_{k=1}^{5} \frac{3}{k^2}$.

Solution

$$\begin{aligned}
\sum_{k=1}^{5} \frac{3}{k^2} &= \frac{3}{1^2} + \frac{3}{2^2} + \frac{3}{3^2} + \frac{3}{4^2} + \frac{3}{5^2} \\
&= 3 + \frac{3}{4} + \frac{1}{3} + \frac{3}{16} + \frac{3}{25} \\
&= \frac{3600}{1200} + \frac{900}{1200} + \frac{400}{1200} + \frac{225}{1200} + \frac{144}{1200} \\
&= \frac{5269}{1200}
\end{aligned}$$

Example 1.25

Determine the sum: $\sum_{k=1}^{6} 5$.

Solution

$$\begin{aligned}
\sum_{k=1}^{6} 5 &= 5 + 5 + 5 + 5 + 5 + 5 \\
&= 5 \times 6 \\
&= 30
\end{aligned}$$

This summation means that the value 5 must be added 6 times, i.e., $a_k = 5$ for all k=1,2,…,6.

To determine how many terms are in a given summation, subtract the bottom index from the top index and add 1. In the summation $\sum_{k=1}^{6} 5$ the number of terms is therefore equal to $N = 6 - 1 + 1 = 6$. Similarly, in the summation $\sum_{t=3}^{12} 4t$ the number of terms is $N = 12 - 3 + 1 = 10$

As suggested in the second example, the index of a summation does not have to start at $k = 1$. It is quite possible to determine the sum $\sum_{t=3}^{12} 4t$, as follows:

Example 1.26

Determine the sum $\sum_{t=3}^{12} 4t$.

SOLUTION:

$$\begin{aligned}\sum_{t=3}^{12} 4t &= 4(3)+4(4)+4(5)+4(6)+4(7)+4(8)+4(9)+4(10)+4(11)+4(12)\\ &= 12+16+20+24+28+32+36+40+44+48\\ &= 300\end{aligned}$$

Just as you should be able to determine the value of a sum written in sigma notation, you should be able to express a series (sum) of terms in sigma notation.

EXAMPLE 1.27

Express $1^2+2^2+3^2+4^2$ **in sigma notation:** $\sum_{k=1}^{4} k^2 = 1^2+2^2+3^2+4^2$.

Note that it is not compulsory to use the letter k to represent the index. Any letter can be used, but letters such as i, j, k, l, m and n are more commonly used. The previous example can therefore also be written as $\sum_{n=1}^{4} n^2 = 1^2+2^2+3^2+4^2$, without changing the meaning of the summation or the value of the sum.

EXAMPLE 1.28

Write the sum $\sqrt{5}+\sqrt{7}+\sqrt{9}+\cdots+\sqrt{21}$ using sigma notation.

SOLUTION

First we need to determine the general term a_k that represents a typical term in the series. It seems as if it should be $a_k = \sqrt{2k-1}$, because all the numbers under the square root signs are positive odd integers. Next, we need to determine the starting and terminating values for the index k. The starting value has to be $k = 3$, as $\sqrt{2k-1} = \sqrt{2(3)-1} = \sqrt{5}$. The terminating value has to be $k = 11$, because $\sqrt{2k-1} = \sqrt{2(11)-1} = \sqrt{21}$.

Hence $\sqrt{5}+\sqrt{7}+\sqrt{9}+\cdots+\sqrt{21} = \sum_{k=3}^{11} \sqrt{2k-1}$.

But there is no unique way to write a sum in sigma notation. The answer we obtained in the previous example is just one way of writing the sum $\sqrt{5}+\sqrt{7}+\sqrt{9}+\cdots+\sqrt{21}$ in sigma notation. It all depends on the choice of index, which in turn will influence how the general term is expressed. We could have chosen the starting value of k to be 4, instead of 3, and obtained $\sqrt{5}+\sqrt{7}+\sqrt{9}+\cdots+\sqrt{21} = \sum_{k=4}^{12} \sqrt{2k-3}$. Note that in this case a different formula for the general term was used.

The index of summation now runs from 4 to 12 and the general term is $a_k = \sqrt{2k-3}$. This does not, however, changes the actual terms of the series, neither the sum of the series. Only a different notation was used for the same series $\sum_{k=3}^{11} \sqrt{2k-1}$.

ASSESSMENT ACTIVITY 1.7

1. Find the first four terms and the 27th term of the sequence represented by the general term $a_k = \frac{(-1)^k}{4k}$.

1.1. What do you notice about the signs of the first four terms?

1.2. Give a possible explanation for your observation.

2. Determine the sum: $\sum_{n=4}^{9} (2n-3)$

3. Without writing them down, determine how many terms are there in the following sum: $\sum_{k=12}^{51} 4k$

4. Represent the following series in sigma notation:

4.1. $2+4+6+8+10+12+14+16$

4.2. $\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}+\frac{7}{8}+\frac{8}{9}+\frac{9}{10}+\frac{10}{11}$

4.3. $3-6+9-12+15-18+21-24+27-30+33-36$

8. Exponents

In this section we recall information regarding integer exponents $\left(a^m\right)$, radicals $\left(\sqrt{a}\right)$ and n-*th* roots $\left(\sqrt[n]{a}\right)$ in order to give meaning to a power $\left(a^{m/n}\right)$ in which the exponent is a fraction.

Integer Exponents

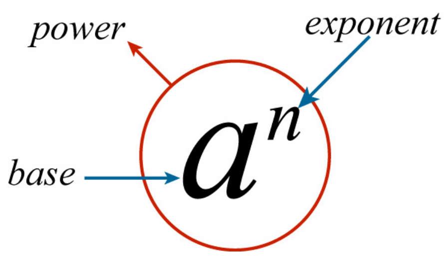
Instead of writing out a product of identical factors, such as $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$, it can be written in exponential form, as 3^5.

Definition of an *n*-th power where *n* is a positive integer

If a is any real number and n is a positive integer, then the nth power of a is

$$a^n = \underbrace{a \cdot a \cdot a \cdot a \cdot \cdots \cdot a \cdot a}_{n \text{ times}}.$$

The number a is called the base and n the exponent.



In order to work with exponents, we need to know the laws according to which they operate.

LAWS OF EXPONENTS

1 $a^m a^n = a^{m+n}$

To multiply powers with the same base, use the same base and add the exponents to obtain the exponent of the product. For example $a^4 a^6 = a^{4+6} = a^{10}$.

Proof: If a is any real number and m and n positive integers, then

$$a^m a^n = (\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})(\underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}) = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a \cdot a}_{m+n \text{ factors}} = a^{m+n}$$

2 $\dfrac{a^m}{a^n} = a^{m-n}$

To divide powers with the same base, use the same base and subtract the exponents (the one in the denominator from the one in the numerator) to obtain the exponent of the quotient. For example, $\frac{a^6}{a^4} = a^{6-4} = a^2$.

Proof: If a is any real number and m and n positive integers, then

$$\frac{a^m}{a^n} = \frac{\overbrace{a \cdot a \cdot a \cdot a \cdot \cdots \cdot a}^{m \text{ factors}}}{\underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}} = \underbrace{a \cdot a \cdot \cdots \cdot a}_{m-n \text{ factors}} = a^{m-n}$$

3 $\left(a^m\right)^n = a^{mn}$

To raise a power to another power, keep the base and multiply the exponents with each other to obtain the exponent of the new power. For example, $\left(a^6\right)^3 = a^{6\times 3} = a^{18}$.

Proof: If a is any real number and m and n are positive integers, then

$$\begin{aligned}\left(a^m\right)^n &= (\underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})^n \\ &= \underbrace{(\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})(\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}}) \cdots (\underbrace{a \cdot a \cdot \cdots \cdot a}_{m \text{ factors}})}_{n \text{ groups of } m \text{ factors}} \\ &= \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a \cdot a}_{mn \text{ factors}} \\ &= a^{mn}\end{aligned}$$

4 $(ab)^n = a^n b^n$

To raise a product to a power, raise each factor in the product to this power and multiply these two powers. For example, $(ab)^4 = a^4 b^4$.

Proof: If a and b are any real numbers and n is a positive integer, then

$$\begin{aligned}(ab)^n &= \underbrace{(ab)(ab)\cdots(ab)}_{n \text{ factors}} \\ &= (\underbrace{a \cdot a \cdot \cdots \cdot a}_{n \text{ factors}}) \cdot (\underbrace{b \cdot b \cdot \cdots \cdot b}_{n \text{ factors}}) \\ &= a^n b^n\end{aligned}$$

5 $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

To raise a quotient to a power, raise both the numerator and the denominator to this power, and find the quotient of these two powers. For example, $\left(\frac{a}{b}\right)^6 = \frac{a^6}{b^6}$.

Proof: If a and b are any real numbers and n is a positive integer, then

$$\left(\frac{a}{b}\right)^n = \underbrace{\frac{a}{b}\cdot\frac{a}{b}\cdot\ldots\cdot\frac{a}{b}}_{n \text{ factors}} = \frac{\overbrace{a\cdot a\cdot a\cdots\cdot a}^{n \text{ factors}}}{\underbrace{b\cdot b\cdot b\cdot\cdots\cdot b}_{n \text{ factors}}} = \frac{a^n}{b^n}$$

There are two important definitions that we must take note of.

- **The definition of a zero exponent**:
  If $a \neq 0$ is any real number then $a^0 = 1$.
  Proof: We want the laws above to hold for zero exponents as well, so $a^0 \cdot a = a^0 \cdot a^1 = a^{0+1} = a^1 = a$. But this is only possible if $a^0 = 1$.

- **The definition of a negative exponent**:
  If $a \neq 0$ is any real number and n is a positive integer, then $a^{-n} = \frac{1}{a^n}$.
  Proof: We want the laws above to hold for negative exponents as well, so $a^n \cdot a^{-n} = a^{n+(-n)} = a^0 = 1$. But this is only possible if $a^{-n} = \frac{1}{a^n}$.

We now conclude with the last two Laws of Exponents.

6 $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

To raise a fraction to a negative power, invert the fraction and change the sign of the exponent to obtain the exponent of the inverted fraction. For example, $\left(\frac{a}{b}\right)^{-2} = \left(\frac{b}{a}\right)^2$.

Proof: From the laws above, we have

$$\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \left(\frac{1}{\frac{a}{b}}\right)^n = \left(\frac{b}{a}\right)^n$$

7 $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$

When moving a power from the numerator to the denominator or vice versa, change the sign of the exponent. For example, $\dfrac{a^{-3}}{b^{-5}} = \dfrac{b^5}{a^3}$.

Proof: By the definition of negative exponents and Property 2 of fractions, we have that $\dfrac{a^{-n}}{b^{-m}} = \dfrac{1/a^n}{1/b^m} = \dfrac{1}{a^n} \times \dfrac{b^m}{1} = \dfrac{b^m}{a^n}$.

Examples 1.29

Eliminate all negative exponents and simplify as far as possible:

1 $x^4x^7 = x^{4+7} = x^{11}$

2 $x^3x^{-9} = x^{3+(-9)} = x^{-6} = \dfrac{1}{x^6}$

3 $\dfrac{y^{12}}{y^5} = y^{12-5} = y^7$

4 $\left(c^2\right)^7 = c^{2\times 7} = c^{14}$

5 $(5x)^2 = 5^2 \cdot x^2 = 25x^2$

6 $\left(\dfrac{h}{3}\right)^3 = \dfrac{h^3}{3^3} = \dfrac{h^3}{27}$

7 $\dfrac{x^{-5}}{y^2} = \dfrac{1}{y^2x^5}$

8
$$\begin{aligned}\left(4a^9c^3\right)^2\left(7ac^6\right) &= \left(4^2\left(a^9\right)^2\left(c^3\right)^2\right)\left(7ac^6\right)\\ &= \left(16a^{18}c^6\right)\left(7ac^6\right)\\ &= (16)(7)a^{18}ac^6c^6\\ &= 112a^{19}c^{12}\end{aligned}$$

9
$$\begin{aligned}\frac{4s^2t^{-3}k^{-6}}{12t^4(sk)^{-4}} &= \frac{1s^2(sk)^4}{3t^4t^3k^6}\\ &= \frac{s^2s^4k^4}{3t^7k^6}\\ &= \frac{s^6}{3t^7k^2}\end{aligned}$$

10 $\left(\dfrac{x}{y}\right)^{-3}\left(\dfrac{\left(y^2x^3\right)^4}{z^{-2}}\right)^5 = \left(\dfrac{y}{x}\right)^3\left(\dfrac{y^8x^{12}}{z^{-2}}\right)^5$

$$= \left(\frac{y}{x}\right)^3\left(y^8x^{12}z^2\right)^5$$

$$= \left(\frac{y^3}{x^3}\right)\left(y^{40}x^{60}z^{10}\right)$$

$$= y^3y^{40}x^{-3}x^{60}z^{10}$$

$$= y^{43}x^{57}z^{10}$$

Radicals (Roots)

After our discussion of powers and exponents, we now fully understand what 2^n means. To give meaning to a power such as $2^{3/4}$, where the exponent is not necessarily an integer but an arbitrary rational number, we need to understand how radicals work. This will be addressed in the following sections.

SQUARE ROOTS

The symbol $\sqrt{\ }$ is used to take (non-negative) square roots. In particular, we write $\sqrt{a} = b$, when $b^2 = a$ where $b \geq 0$ and $a \geq 0$. To take the square root of a non-negative number a therefore means that we have to find a non-negative number b such that, when it is multiplied by itself, the answer is a. For instance, $\sqrt{16} = 4$, because $4^2 = 16$, and $4 \geq 0$.

The symbol $-\sqrt{\ }$ is used to get hold of the negative of the square root of a (non-negative) number. So, if $\sqrt{a} = b$, where $b \geq 0$ and $a \geq 0$, then $-\sqrt{a} = -b$. For instance, $-\sqrt{16} = -4$. In words it means that the negative of the square root of 16 is equal to negative 4.

n TH ROOTS

Square roots are special cases of the more general nth roots. The nth root of a number x is the number that, when raised to the nth power, gives x.

Definition of an nth root

If n is any positive integer, then the nth root of a is defined as follows:

$\sqrt[n]{a} = b$ if and only if $b^n = a$

If n is even, we must have that $a \geq 0$ and $b \geq 0$. In case n is odd, a could be any real number. Also note that b always has the same sign as a.

EXAMPLES 1.30

1 $\sqrt[4]{625} = 5$, because $5^4 = 625$

2 $\sqrt[3]{-64} = -4$, because $(-4)^3 = -64$

It is important to note that, when n is even, and $\sqrt[n]{a} = b$, then we have that $b^n = a$ and $(-b)^n = a$, although we only choose the non-negative b as the unique n-th root of a. If we really want to consider the negative root, we simply write $-\sqrt[n]{a}$. If n is odd, we do not have this difficulty, since in this case there is only one real number b such that $b^n = a$.

PROPERTIES OF n TH ROOTS

The properties of n th roots are now stated (with examples to illustrate these properties). It is understood that all roots exist. (For example, if n is even, then $\sqrt[n]{a}$ doesn't exist if $a < 0$.) Also remember that $\sqrt{a} = \sqrt[2]{a}$.

	Property	Example
1	$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$	$\sqrt[3]{(-8)(27)} = \left(\sqrt[3]{-8}\right)\left(\sqrt[3]{27}\right) = (-2)(3) = -6$
2	$\sqrt[n]{\dfrac{a}{b}} = \dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}$	$\sqrt[4]{\dfrac{81}{16}} = \dfrac{\sqrt[4]{81}}{\sqrt[4]{16}} = \dfrac{3}{2}$
3	$\sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}$	$\sqrt[3]{\sqrt{46656}} = \sqrt[6]{46656} = 6$
4	$\sqrt[n]{a^n} = a$, if n is odd	$\sqrt[5]{(-2)^5} = -2$ and $\sqrt[3]{2^3} = 2$
5	$\sqrt[n]{a^n} = \lvert a\rvert$, if n is even	$\sqrt[4]{(-3)^4} = \lvert -3\rvert = 3$ because $\sqrt[4]{(-3)^4} = \sqrt[4]{81} = 3$ and $\sqrt[4]{3^4} = \sqrt[4]{81} = 3$. The answer must be non- negative, since 4 is even.

EXAMPLES 1.31

Simplify the following expressions

1 $\sqrt[3]{x^7}$

2 $\sqrt[4]{256x^8y^5}$

Combine the following radicals and simplify as far as possible:

3 $\sqrt{72} + \sqrt{162}$

4 $\sqrt[3]{8a^8} - \sqrt[3]{-27a^5}$

SOLUTIONS

1 $$\sqrt[3]{x^7} = \sqrt[3]{x^3x^3x}$$
$$= \left(\sqrt[3]{x^3}\right)\left(\sqrt[3]{x^3}\right)\left(\sqrt[3]{x}\right)$$

$$= (x)(x)(\sqrt[3]{x})$$
$$= x^2(\sqrt[3]{x})$$

2 $\sqrt[4]{256x^8y^5} = \sqrt[4]{256}(\sqrt[4]{x^8})(\sqrt[4]{y^5})$
$$= (\sqrt[4]{4^4})(\sqrt[4]{x^4x^4})(\sqrt[4]{y^4y})$$
$$= (\sqrt[4]{4^4})(\sqrt[4]{x^4})(\sqrt[4]{x^4})(\sqrt[4]{y^4})(\sqrt[4]{y})$$
$$= |4||x||x||y|(\sqrt[4]{y})$$
$$= 4x^2|y|(\sqrt[4]{y})$$

3 $\sqrt{72} + \sqrt{162} = \sqrt{36(2)} + \sqrt{81(2)}$
$$= \sqrt{36}\sqrt{2} + \sqrt{81}\sqrt{2}$$
$$= 6\sqrt{2} + 9\sqrt{2}$$
$= (6+9)\sqrt{2}$ *(Take $\sqrt{2}$ out as a common factor.)*
$$= 15\sqrt{2}$$

4 $\sqrt[3]{8a^8} - \sqrt[3]{-27a^5} = \sqrt[3]{2^3(a^2)^3a^2} - \sqrt[3]{(-3)^3a^3a^2}$
$$= (\sqrt[3]{2^3})(\sqrt[3]{(a^2)^3})(\sqrt[3]{a^2}) - (\sqrt[3]{(-3)^3})(\sqrt[3]{a^3})(\sqrt[3]{a^2})$$
$$= 2a^2(\sqrt[3]{a^2}) - (-3)a(\sqrt[3]{a^2})$$
$= a(\sqrt[3]{a^2})(2a+3)$ *(Take $a(\sqrt[3]{a^2})$ out as a common factor).*

Rational Exponents

Now that we understand how radicals work, we can give meaning to a power such as $2^{3/4}$, where the exponent is an arbitrary rational number.

Consider the symbol $a^{1/n}$. If we want the laws of exponents to hold for rational exponents as well, then we have that:

$$\left(a^{1/n}\right)^n = a^{(1/n)n} = a^{n/n} = a^1 = a \qquad \text{(i)}$$

Now recall the definition of an nth root which says that if n is any positive integer, then the nth root of a is defined as follows:

$$\sqrt[n]{a} = b \qquad \text{means that } b^n = a \qquad \text{(ii)}$$

So, if we let $b = a^{1/n}$, then, from (i) and (ii), we must have that:

$$\sqrt[n]{a} = b = a^{1/n}.$$

Definition of rational exponents

For any rational exponent m/n where m and n are integers that do not share any common prime factors, and $n > 0$, we define:

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m \text{ or, equivalently, } a^{m/n} = \sqrt[n]{a^m}$$

If n is even, it is required that $a \geq 0$.

It can be shown that all the Laws of Exponents also hold for rational exponents. This is left as an exercise.

Examples 1.32

Simplify the following expressions as far as possible by using the definition of rational exponents and the laws of exponents.

1. $16^{3/2}$
2. $8^{-1/3}$
3. $\dfrac{x^{1/4}x^{1/2}}{x}$
4. $\left(2a^2b^6c^4\right)^{3/2}$
5. $\sqrt[3]{x^2\sqrt{x}}$

Solutions

1 $16^{3/2} = \left(\sqrt{16}\right)^3 = 4^3 = 64$

2 $8^{-1/3} = \dfrac{1}{8^{1/3}} = \dfrac{1}{\sqrt[3]{8}} = \dfrac{1}{2}$

3 $\dfrac{x^{1/4}x^{1/2}}{x} = \dfrac{x^{1/4}x^{2/4}}{x^{4/4}} = x^{1/4+2/4-4/4} = x^{-1/4} = \dfrac{1}{x^{1/4}}$

It is customary in mathematics to write negative exponents in terms of their positive equivalents in the final answer.

4 $\left(2a^2b^6c^4\right)^{3/2} = 2^{3/2}\left(a^2\right)^{3/2}\left(b^6\right)^{3/2}\left(c^4\right)^{3/2}$

$= 2^{\left(2/2+1/2\right)}\left(a^{6/2}\right)\left(b^{18/2}\right)\left(c^{12/2}\right)$

$= \left(2^1\right)\left(2^{1/2}\right)a^3b^9c^6$

$= 2\sqrt{2}a^3b^9c^6$

5 $\sqrt[3]{x^2\sqrt{x}} = \sqrt[3]{x^2x^{1/2}}$

$= \left(\sqrt[3]{x^2}\right)\left(\sqrt[3]{x^{1/2}}\right)$

$$= \left(x^2\right)^{1/3}\left(x^{1/2}\right)^{1/3}$$
$$= x^{2/3}x^{1/6}$$
$$= x^{2/3+1/6}$$
$$= x^{5/6}$$

RATIONALIZING THE DENOMINATOR

It is often useful to eliminate the radical in the denominator of a fraction. This is done by multiplying both the numerator and the denominator of the fraction by an appropriate expression containing the radical. This procedure is called rationalizing the denominator. To rationalize the denominator of the fraction $\frac{1}{\sqrt{a}}$, we multiply both the numerator and the denominator by $\sqrt{a}$. So the fraction is multiplied by $\frac{\sqrt{a}}{\sqrt{a}} = 1$. Its value, therefore, stays unchanged, but its appearance changed to a more convenient form.

So, $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{a}} \cdot \frac{\sqrt{a}}{\sqrt{a}} = \frac{\sqrt{a}}{\left(\sqrt{a}\right)^2} = \frac{\sqrt{a}}{a}$.

EXAMPLES 1.33

Rationalize the denominators of the following expressions:

1 $\frac{3}{\sqrt{5}}$

2 $\frac{1}{\sqrt[3]{x^5}}$

SOLUTIONS

1 $\frac{3}{\sqrt{5}} = \frac{3}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{3\sqrt{5}}{\left(\sqrt{5}\right)^2} = \frac{3\sqrt{5}}{5}$

2 $\frac{1}{\sqrt[3]{x^5}} = \frac{1}{x^{5/3}} = \frac{1}{x \cdot x^{2/3}} = \frac{x^{1/3}}{x \cdot x^{2/3} \cdot x^{1/3}} = \frac{\sqrt[3]{x}}{x^2}$

In the second example we chose to multiply by $\frac{\sqrt[3]{x}}{\sqrt[3]{x}}$, because there was a factor $x^{2/3}$ in the denominator. Multiplying this by the cube root of x gives an integral power of x in the denominator.

ASSESSMENT ACTIVITY 1.8

1. Evaluate $\dfrac{\sqrt{48}}{\sqrt{3}}$

2. Simplify the expression $\sqrt[3]{54}-\sqrt[3]{16}$

3. Simplify the expressions and eliminate any negative exponents.

3.1. $\dfrac{a^{-3}b^{4}}{a^{-5}b^{5}}$

3.2. $\left(3ab^{2}c\right)\left(\dfrac{2a^{2}b}{c^{3}}\right)^{-2}$

3.3. $\left(4x^{6}y^{8}\right)^{3/2}$

3.4. $\left(\dfrac{a^{2}b^{-3}}{x^{-1}y^{2}}\right)^{3}\left(\dfrac{x^{-2}b^{-1}}{a^{3/2}y^{1/3}}\right)$

4. Simplify the expressions.

4.1. $\left(\sqrt[3]{a^{2}b}\right)\left(\sqrt[3]{a^{4}b}\right)$

4.2. $\sqrt[3]{\sqrt{64x^{6}}}$

5. Rationalize the denominators in the following expressions.

5.1. $\sqrt{\dfrac{x^{5}}{2}}$

5.2. $\dfrac{1}{\sqrt[7]{x^{3}}}$

9. Logarithms

We need logarithms to solve equations such as $b^x = a$ for x. The variable that we want to solve for appears in the exponent. To solve for x we need to be able to express it in terms of a and b. We introduce a new notation to help us to solve this kind of equation. The *logarithm* of a to the base b, denoted by $\log_b a$, is the real number x such that $b^x = a$

Logarithm Notation

If $a > 0$ and $b > 0,\ b \neq 1,$ then $\log_b a = x$ if and only if $b^x = a$.

The equation $x = \log_b a$ is read as "x is the logarithm of a to the base b" or "x is equal to log base b of a". Note that the logarithm is equal to x, while x is also the exponent in the equation $b^x = a$. Thus, a logarithm is an exponent. In this case, x is the exponent to which b must be raised to produce a.

The number a is obtained by raising the positive base b to some power, namely x. Consequently, a is always positive in the equation $b^x = a$. So, a has to be positive in the equation $x = \log_b a$. In other words, $\log_b a$ is defined only for $a > 0$.

The logarithm of a negative number (or zero) is not defined.

Here are some examples of equations written in logarithmic form along with their equivalents in exponential form.

LOGARITHMIC EQUATION	**EXPONENTIAL EQUATION**
$\log_3 9 = 2$	$3^2 = 9$
$\log_2 8 = 3$	$2^3 = 8$
$\log_4 1 = 0$	$4^0 = 1$
$\log_{10} 0{,}01 = -2$	$10^{-2} = 0{,}01$
$\log_e 1 = 0$	$e^0 = 1$
$\log_{10}(-2) =$ undefined	$10^? = -2$ There is no power of 10 equal to -2

The two most popular bases for logarithms are 10 and e. Logarithms with base 10 are called common logarithms and $\log_{10} x$ is usually simply written as $\log x$. Hence, when no base is shown, it is assumed that the base is 10. Logarithms with base $e = 2{,}7182818...$ are called natural logarithms and $\log_e x$ is sometimes written as $\ln x$. The number

$e = 2{,}7182818\ldots$ is a famous constant (such as π) and is used, for example, as a base in certain natural growth processes.

EXAMPLES 1.34

Solve the following equations for x or y:

1 $\log_2 64 = y$

2 $\log_3\left(\frac{1}{243}\right) = y$

3 $\log_{49} x = -\frac{1}{2}$

4 $\ln x = 4$

5 $\log_{\frac{1}{16}} x = \frac{1}{4}$

SOLUTIONS

1 $2^y = 64 = 2^6$, therefore $y = 6$

2 $3^y = \frac{1}{243} = \frac{1}{3^5} = 3^{-5}$, therefore $y = -5$

3 $x = 49^{-1/2} = \frac{1}{49^{1/2}} = \frac{1}{\sqrt{49}} = \frac{1}{7}$

4 $x = e^4 \approx 54{,}598$

5 $x = \left(\frac{1}{16}\right)^{1/4} = \left(\frac{1}{2^4}\right)^{1/4} = \left(2^{-4}\right)^{1/4} = 2^{-1} = \frac{1}{2}$

The Laws of Logarithms

From our current knowledge of logarithms we have, for example, the following result:

$$\log_2 8 + \log_2 16 = \log_2 2^3 + \log_2 2^4 = 3 + 4 = 7 = \log_2 128 = \log_2 (8)(16)$$

Therefore $\log_2(8 \cdot 16) = \log_2 8 + \log_2 16$. It turns out that this equation is a special case of the first law of logarithms. Here is a list of the important laws of logarithms:

Laws of Logarithms

If M and N are positive, $b > 0$, and $b \neq 1$, then

(i) $\log_b (MN) = \log_b M + \log_b N$

(ii) $\log_b \left(\frac{M}{N}\right) = \log_b M - \log_b N$

(iii) $\log_b\left(N^k\right) = k\log_b N$

(iv) $\log_b(b^x) = x$

(v) $b^{\log_b x} = x$

In addition, $\log 1 = 0$, because $10^0 = 1$ and, in general, $\log_b 1 = 0$ for any base b.

Examples 1.35

Use the laws of logarithms to expand the following expressions:

1 $\log\left(\frac{x+2}{x+5}\right)$

2 $\log 3^{2x}$

3 $\log\left(e^3 x\right)$

4 $\log_b\left(\frac{3}{2x}\right)$

Use the laws of logarithms to simplify the following expressions:

5 $\log 3 + \log x$

6 $\log x - \log 2$

7 $2\log_b(x-1) + \frac{1}{2}\log_b x$

Solutions

1 Use property (ii): $\log\left(\frac{x+2}{x+5}\right) = \log(x+2) - \log(x+5)$

(provided that $x + 2 > 0$ and $x + 5 > 0$)

2 Use property (iii): $\log 3^{2x} = 2x\log 3$

3 Use property (i): $\log\left(e^3 x\right) = \log e^3 + \log x = 3\log e + \log x$

(provided that $x > 0$)

4 Use properties (i) and (ii):

$$\begin{aligned}\log_b\left(\frac{3}{2x}\right) &= \log_b 3 - \log_b 2x \\ &= \log_b 3 - \left(\log_b 2 + \log_b x\right) \\ &= \log_b 3 - \log_b 2 - \log_b x\end{aligned}$$

(provided that $x > 0$)

5 Use property (i): $\log 3 + \log x = \log 3x$ (provided that $x > 0$)

6 Use property (ii): $\log x - \log 2 = \log\left(\frac{x}{2}\right)$ (provided that $x > 0$)

7 Use properties (iii) and (i):
$$2\log_b(x-1)+\frac{1}{2}\log_b x=\log_b(x-1)^2+\log_b x^{1/2}$$
$$=\log_b(x-1)^2\left(x^{1/2}\right)$$
(provided that $x>0$)

ASSESSMENT ACTIVITY 1.9

Solve the following equations for x, y or b:

1 $y=\log_9 27$

2 $\log_b 8=\dfrac{3}{4}$

3 $\log_8 x=-\dfrac{2}{3}$

4 $y=\log_{27} 3$

5 $\log_b\left(\dfrac{1}{128}\right)=-7$

Use the properties of logarithms to simplify each of the following expressions:

6 $\log_b x+\log_b x+\log_b 3$

7 $\log x-\log(x-1)-2\log(x-2)$

8 $\log 4-\log 10+2$

9 $\dfrac{\log A^2-\log A}{\log B-\frac{1}{2}\log B}$

10 $2\log A-3\log B-\dfrac{\log A}{2}$

10. End of section comments

This wraps up all the content that you had to acquire before being able to proceed to the next sections. Make sure that you understand and can confidently work with everything that was addressed in this section. We will now proceed to the next section which is on calculator usage.

Feedback

The answers to all the activities can be found here:

Learning activity 1.2

1. $\frac{1}{3}$

2. $\frac{28}{31}$

3. $\frac{7}{4}$

4. $\frac{31}{20}$

5. $-\frac{1}{30}$

Learning activity 1.3

1.

- **1.1.** $2{,}4\dot{5}$
- **1.2.** $1{,}9\overline{78}$
- **1.3.** $0{,}2\overline{12}$

2. $x = \frac{592176}{99900}$

Learning activity 1.4

1.

- **1.1.** 7
- **1.2.** 6,9
- **1.3.** 6,91
- **1.4.** 6,910
- **1.5.** 6,9103
- **1.6.** 6,91028
- **1.7.** 6,910284
- **1.8.** 6,9102837
- **1.9.** 6,91028375

1.10. $6{,}910283747$

2. The last approximation $6{,}910283747$ is the closest. There is only $0{,}0000000005$ difference between this approximation and the original value.

Learning activity 1.5

1. $45{,}6\%$
2. $76{,}47\%$
3. Decimal: $0{,}38$

Fraction: $\frac{38}{100}$

4. Physics
5. $R10199{,}15$

6. 8,57%

7. 17,86%

8. B had the biggest increase, namely 100%

Learning activity 1.6

1. $15:13$
2. 16-year old daughter: $R10200{,}65$
13-year old daughter: $R8288{,}03$
8-year old daughter: $R5100{,}32$
3. $R66{,}67$ for 8 buns
4.
4.1. 20 days
4.2. 5 days

Assessment activity 1.7

1.

1.1. $a_1 = -\frac{1}{4}$, $a_2 = \frac{1}{8}$, $a_3 = -\frac{1}{12}$, $a_4 = \frac{1}{16}$, $a_{27} = -\frac{1}{108}$

1.2. The signs alternate.

1.3. (-1) raised to an odd power is negative and (-1) raised to an even power is positive.

2. 60

3. 40 terms

4.

4.1. $\sum_{n=1}^{8} 2n$

4.2. $\sum_{n=2}^{10} \frac{n}{n+1}$

4.3. $\sum_{n=1}^{12} 3n$

Assessment activity 1.8

1. $\sqrt[3]{2}$
2.
   2.1. $\dfrac{a^2}{b}$
   2.2. $\dfrac{3c^7}{4a^3}$
   2.3. $8x^9y^{12}$
   2.4. $\dfrac{a^{9/2}x}{b^{10}y^{19/3}}$
3.
   3.1. $a^2b^{2/3}$
   3.2. $2x$
4.
   4.1. $\dfrac{\sqrt{2x^5}}{2}$
   4.2. $\dfrac{\sqrt[7]{x^4}}{x}$

Assessment activity 1.9

1 $y = \dfrac{3}{2}$

2 $b = 16$

3 $x = \dfrac{1}{4}$

4 $y = \dfrac{1}{3}$

5 $b = 2$

6 $\log_b\left(3x^2\right)$

7 $\log\left(\dfrac{x}{(x-1)(x-2)^2}\right)$

8 $\log\left(\dfrac{2}{5}\right)+2$ or $2\log 2+1$

9 $\dfrac{2\log A}{\log B}$

10 $\log\left(\dfrac{A^{3/2}}{B^3}\right)$

Tracking my progress

You have reached the end of this section. Check whether you have achieved the learning outcomes for this section.

LEARNING OUTCOMES	✓ I FEEL CONFIDENT	✓ I DON'T FEEL CONFIDENT
Round off decimals to a given number of places		
Multiply and divide fractions		
Add or subtract fractions with the same or different denominators		
Simplify fractions		
Convert a fraction to a decimal by using a calculator		
Use the correct notation to indicate a recurring decimal		
Convert a decimal to a fraction		
Round off decimals to a given number of places		
Convert decimals and fractions to percentages and vice versa		
Do simple increasing and decreasing problem solving by using percentages		
Do simple problem solving by using direct and inverse proportion		
Write a given series in sigma notation		
Know and be able to apply all the Laws of Exponents to simplify expressions		
Know and apply the laws of logarithms to expand a logarithmic expression		

Now answer the following questions honestly:

1 What did you like best about this section?

2 What did you find most difficult in this section?

3 What do you need to improve on?

4 How will you do this?